Série 6

Ex2 $\Sigma_1 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = z^2, \ 0 \leq z \leq 1\}$

coordonnées cylindriques

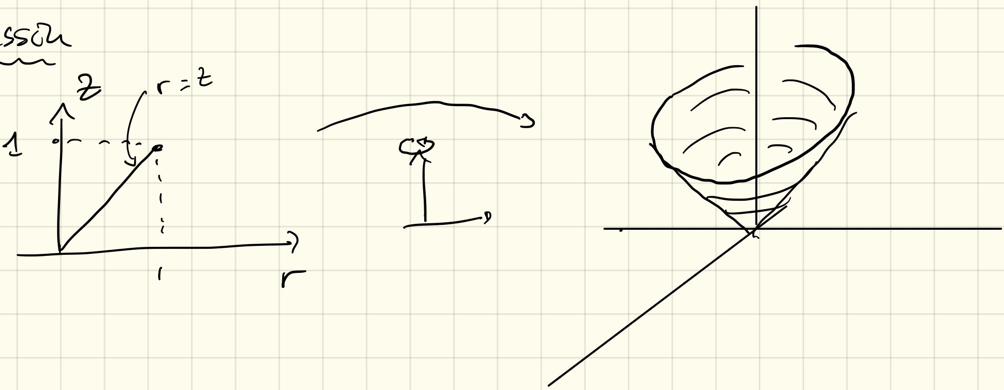
$x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $r \geq 0$, $\theta \in [0, 2\pi]$, $z \in \mathbb{R}$.

$x^2 + y^2 = z^2 \iff r^2 = z^2 \overset{r \geq 0}{\underset{z \geq 0}{\iff}} r = z$.

Paramétrisation Σ_1 :

$\sigma(r,\theta) = (r\cos\theta, r\sin\theta, r)$, $r \in [0,1]$, $\theta \in [0, 2\pi]$

Dessin



$$\sigma_r(r,\theta) = (\cos\theta, \sin\theta, 1)$$

$$\sigma_\theta(r,\theta) = (-r\sin\theta, r\cos\theta, 0)$$

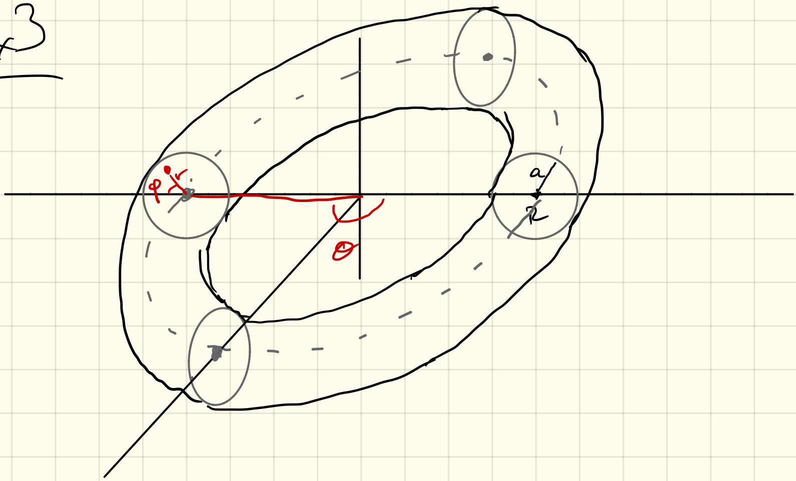
$$\sigma_r \wedge \sigma_\theta = \begin{vmatrix} e_1 & e_2 & e_3 \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \\ r \end{pmatrix}$$

(on obtient l'opposé de ce qu'il y a dans le livre, parce qu'on a renversé l'ordre de σ_r $(=\sigma_z)$ et σ_θ)

$$\Rightarrow |\sigma_r \wedge \sigma_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2}\,r$$

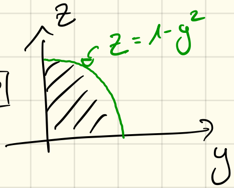
Ex3

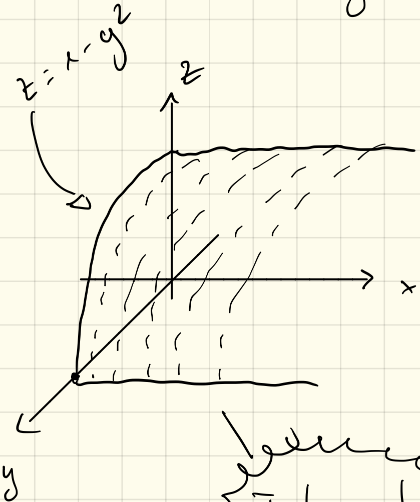


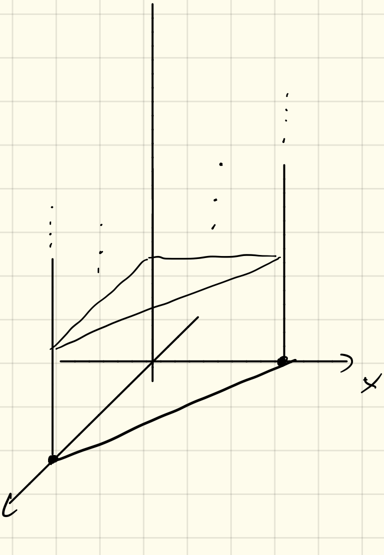
Ex 4 3)

$D = \{ (x,y,z) \in \mathbb{R}^3 : x,y,z \geq 0, \quad x+y \leq 1, \quad z \leq 1-y^2 \}$

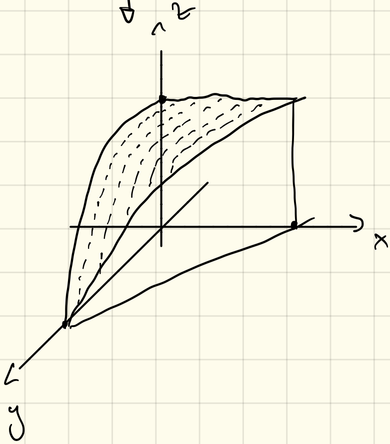
$\boxed{x=0}$







Intersection



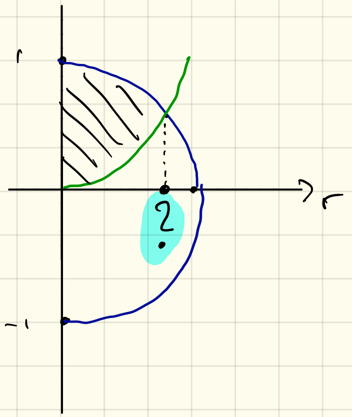
Ex 5 1)

$$D = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1,\ x^2+y^2 \leq z\}$$

symétrie sphérique — symétrie cylindrique

On utilise les coordonnées cylindriques,

$$r^2+z^2 \leq 1, \quad r^2 \leq z$$



$r^2+z^2=1$

$r^2=z$

$$\begin{cases} r^2+z^2=1 \\ r^2=z \end{cases} \Rightarrow z+z^2=1 \Rightarrow z^2+z-1=0$$

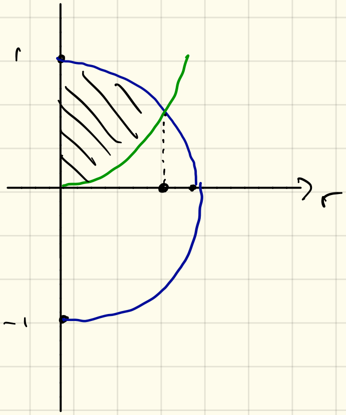
$$\Rightarrow z = \frac{-1\pm\sqrt{1+4}}{2} = \begin{cases} -\frac{1-\sqrt{5}}{2} \\ \frac{\sqrt{5}-1}{2} \end{cases}$$

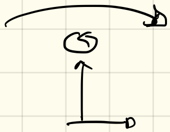
$$r^2=z \Rightarrow r=\sqrt{\frac{\sqrt{5}-1}{2}}$$

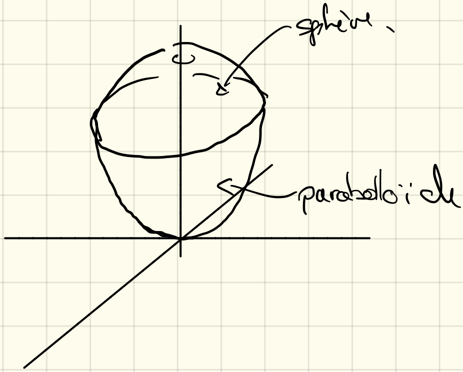
$\left(r^2=\frac{-1-\sqrt{5}}{2}\right.$ n'a pas de sol car $\left.\frac{-1-\sqrt{5}}{2}<0\right)$

On arrive à $r\in\left[0,\sqrt{\frac{\sqrt{5}-1}{2}}\right]$, $z\in\left[r^2,\sqrt{1-r^2}\right]$

ou $z\in\left[0,\frac{\sqrt{5}-1}{2}\right]$, $r\in\left[0,\sqrt{z}\right]$ + $z\in\left[\frac{\sqrt{5}-1}{2},1\right]$, $r\in\left[0,\sqrt{1-z^2}\right]$







$$x = r\cos\theta,\quad y = r\sin\theta,\quad z = z,\quad r \in \left[0, \sqrt{\frac{\sqrt{5}-1}{2}}\right],\quad z \in \left[r^2, \sqrt{1-r^2}\right]$$

$$\iiint_D 1 \, dx\,dy\,dz = \int_0^{2\pi} \int_0^{\sqrt{\frac{\sqrt{5}-1}{2}}} \int_{r^2}^{\sqrt{1-r^2}} r \, dz\, dr\, d\theta$$

$$= 2\pi \int_0^{\sqrt{\frac{\sqrt{5}-1}{2}}} r\sqrt{1-r^2} - r^3 \, dr = 2\pi \left[-\frac{1}{3}(1-r^2)^{3/2} - \frac{1}{4} r^4 \right]_0^{\sqrt{\frac{\sqrt{5}-1}{2}}}$$

$$= 2\pi \left(-\frac{1}{3}\left(1 - \frac{\sqrt{5}-1}{2}\right)^{3/2} - \frac{1}{4}\left(\frac{\sqrt{5}-1}{2}\right)^2 + \frac{1}{3} \right)$$

$$= 2\pi \left(\frac{1}{3} - \frac{1}{3}\left(\frac{3-\sqrt{5}}{2}\right)^{3/2} - \frac{1}{16}\left(6 - 2\sqrt{5}\right) \right)$$

$$= \frac{\pi}{12}\left(8 - 8\left(\frac{3-\sqrt{5}}{2}\right)^{3/2} - 3(3-\sqrt{5})\right)$$

$$= \frac{\pi}{12}\left(3\sqrt{5} - 1 - 8\left(\frac{3-\sqrt{5}}{2}\right)^{3/2}\right)$$

On pourrait s'arrêter là.

Mais :

$$\left(\frac{3-\sqrt{5}}{2}\right)^3 = \frac{27 - 27\sqrt{5} + 45 - 5\sqrt{5}}{8}$$

$$= \frac{72 - 32\sqrt{5}}{8} = 9 - 4\sqrt{5} = 2^2 - 2\cdot 2\sqrt{5} + (\sqrt{5})^2$$

$$= (2-\sqrt{5})^2.$$

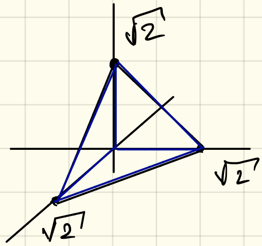
Ainsi, $\iiint_D 1\,dx\,dy\,dz = \frac{\pi}{12}\left(3\sqrt{5} - 1 - 8|2-\sqrt{5}|\right)$

$\sqrt{5} > 2$

$$= \frac{\pi}{12}\left(3\sqrt{5} - 1 - 8\sqrt{5} + 16\right)$$

$$= \frac{\pi}{12}\left(15 - 5\sqrt{5}\right) = \frac{5\pi}{12}\left(3-\sqrt{5}\right)$$

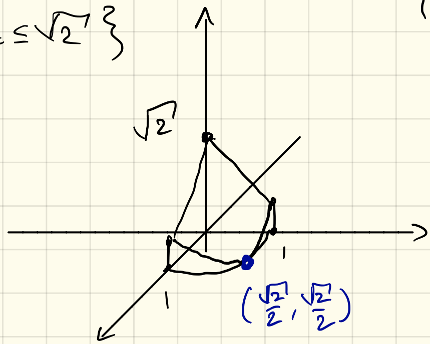
2) $D=\{(x,y,z)\in\mathbb{R}^3 : x,y,z\geq 0,\ x+y+z\leq\sqrt{2},\ x^2+y^2\leq 1\}$.



$\{(x,y,z)\in\mathbb{R}^3 :$
$x,y,z\geq 0,\ x+y+z\leq\sqrt{2}\}$

Intersection

$\{(x,y,z)\in\mathbb{R}^3 : x,y,z\geq 0,$
$x^2+y^2\leq 1\}$



Coordonnées cylindriques: $x=r\cos\theta,\ y=r\sin\theta,\ z=z$

$x\geq 0,\ y\geq 0 \iff \theta\in[0,\pi/2]$	$z\geq 0,\ x+y+z\leq\sqrt{2}$
$x^2+y^2\leq 1 \iff r\in[0,1]$	$\Updownarrow$
	$z\in[0,\sqrt{2}-r(\cos\theta+\sin\theta)]$

$$\iiint_D 1\,dx\,dy\,dz=\int_0^{\pi/2}\int_0^1\int_0^{\sqrt{2}-r(\cos\theta+\sin\theta)} r\,dz\,dr\,d\theta$$